P.G. Sem-2nd
paper-VI, unit-1st: complex integration
Dr. G.D. Singh

Theorem 6. Laurent's Theorem

(1) If $f(z)$ is analytic in the closed ring bounded by two concentric circles C and C' of centre a and ~~radius~~ radii R and R' $(R' < R)$. If z is any pt of the annulus then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

where $a_n = \dfrac{1}{2\pi i} \displaystyle\int_C \dfrac{f(t)\,dt}{(t-a)^{n+1}}$

$$b_n = \frac{1}{2\pi i} \int_{C'} \frac{f(t)\,dt}{(t-a)^{-n+1}}$$

2 गुरु

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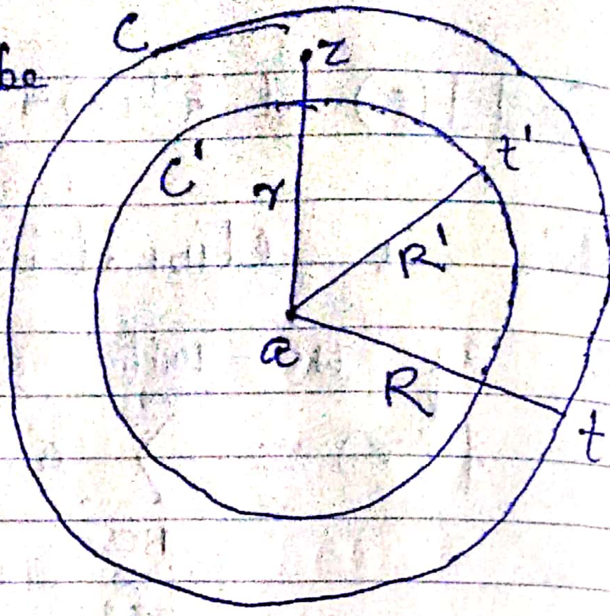
Proof :→ Let $f(z)$ be analytic in the closed ring bounded by two concentric circles C and C' of centre a and radii R and R' $(R' < R)$. If z is any pt within the ring space, then $R' < |z-a| < R$



By extension to Cauchy's integral formula, [By cauchy's integral formula for multi-connected region we have]

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(t)\,dt}{t-z} - \frac{1}{2\pi i}\int_{C'} \frac{f(t)}{t-z}\,dt$$

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$$= \frac{1}{2\pi i}\int_C \frac{f(t)\,dt}{(t-a)-(z-a)} + \frac{1}{2\pi i}\int_{C'} \frac{f(t)}{z-t}\,dt$$

$$= \frac{1}{2\pi i}\int_C \frac{f(t)\,dt}{(t-a)-(z-a)} + \frac{1}{2\pi i}\int_{C'} \frac{f(t)\,dt}{(z-a)-(t-a)}$$

$$= \frac{1}{2\pi i}\int_C \frac{f(t)}{t-a}\left\{\frac{1}{1-\frac{z-a}{t-a}}\right\}dt$$

$$+ \frac{1}{2\pi i}\int_{C'} \frac{f(t)}{z-a}\left\{\frac{1}{1-\frac{t-a}{z-a}}\right\}dt$$

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$$= \frac{1}{2\pi i}\int_C \frac{f(t)}{(t-a)}\left\{1-\frac{z-a}{t-a}\right\}^{-1} dt$$

$$+ \frac{1}{2\pi i}\int_{C'} \frac{f(t)}{z-a}\left\{1-\frac{t-a}{z-a}\right\}^{-1} dt$$

$$= \frac{1}{2\pi i}\int_C \frac{f(t)}{t-a}\left\{1+\frac{z-a}{t-a}+\frac{(z-a)^2}{(t-a)^2}+\cdots\right\} dt$$

$$+ \frac{1}{2\pi i}\int_{C'} \frac{f(t)}{z-a}\left\{1+\frac{t-a}{z-a}+\frac{(t-a)^2}{(z-a)^2}+\cdots\right\} dt$$

$$= \frac{1}{2\pi i}\int_C \frac{f(t)}{t-a}\left\{1+\frac{z-a}{t-a}+\frac{(z-a)^2}{(t-a)^2}+\cdots + \left(\frac{z-a}{t-a}\right)^n + \left(\frac{z-a}{t-a}\right)^{n+1}\left\{1-\frac{z-a}{t-a}\right\}^{-1}\right\} dt$$

$$+ \frac{1}{2\pi i}\int_{C'} \frac{f(t)}{z-a}\left\{1+\frac{t-a}{z-a}+\left(\frac{t-a}{z-a}\right)^2+\cdots + \left(\frac{t-a}{z-a}\right)^n + \left(\frac{t-a}{z-a}\right)^{n+1}\left\{1-\frac{t-a}{z-a}\right\}^{-1}\right\} dt$$

$$\Big[\because (1-b)^{-1} = 1+b+b^2+\cdots+b^n+b^{n+1}+b^{n+2}+\cdots$$

$$= 1+b+b^2+\cdots+b^n+b^{n+1}(1+b+b^2+\cdots)$$

$$= 1+b+b^2+\cdots+b^n+b^{n+1}(1-b)^{-1}\Big]$$

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Let us take

$$\left.\begin{aligned} a_n &= \frac{1}{2\pi i}\int_C \frac{f(t)\,dt}{(t-a)^{n+1}} \\ b_n &= \frac{1}{2\pi i}\int_{C'} \frac{f(t)\,dt}{(t-a)^{-n+1}} = a_{-n} \end{aligned}\right\} \quad (A)$$

$$\therefore f(z) = \left[\frac{1}{2\pi i}\int_C \frac{f(t)}{t-a}\,dt + \frac{z-a}{2\pi i}\int_C \frac{f(t)\,dt}{(t-a)^2} + \frac{(z-a)^2}{2\pi i}\int_C \frac{f(t)\,dt}{(t-a)^3} + \cdots \right.$$

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$$\left. + \frac{(z-a)^n}{2\pi i}\int_C \frac{f(t)\,dt}{(t-a)^{n+1}} + \frac{(z-a)^{n+1}}{2\pi i}\int_C \frac{f(t)\,dt}{(t-a)^{n+1}\left\{1-\frac{z-a}{t-a}\right\}}\right]$$

$$+ \left[\frac{1}{2\pi i}\int_{C'} \frac{f(t)\,dt}{z-a} + \frac{t-a}{2\pi i}\int_{C'} \frac{f(t)\,dt}{(z-a)^2} + \frac{(t-a)^2}{2\pi i}\int_{C'} \frac{f(t)\,dt}{(z-a)^3} + \cdots \right.$$

$$\left. + \frac{(t-a)^n}{2\pi i}\int_{C'} \frac{f(t)\,dt}{(z-a)^{n+1}} + \frac{(t-a)^{n+1}}{2\pi i}\int_{C'} \frac{f(t)\,dt}{(z-a)^{n+2}\left\{1-\frac{t-a}{z-a}\right\}}\right]$$

(B.)

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On account of (A)

$$f(z) = \left[a_0 + (z-a)a_1 + (z-a)^2 a_2 + \cdots + a_n(z-a)^n + U_{n+1}\right]$$

$$+ \left[\frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \cdots + \frac{b_n}{(z-a)^n} + V_{n+1}\right] \quad (C)$$

where $U_{n+1} = \frac{1}{2\pi i}\int_C \frac{f(t)}{t-z}\left(\frac{z-a}{t-a}\right)^{n+1} dt$

$$V_{n+1} = \frac{1}{2\pi i}\int_{C'} \frac{f(t)\,dt}{z-t} \times \left(\frac{t-a}{z-a}\right)^{n+1} dt$$

Let $M = \max|f(t)|$ on C

$M' = \max|f(t)|$ on C'

$$|U_{n+1}| \le \frac{1}{2\pi}\int_C |f(t)|\left|\frac{z-a}{t-a}\right|^{n+1} \frac{|dt|}{|t|-|z|}$$

$$\le \frac{M}{2\pi}\left(\frac{r}{R}\right)^{n+1} \frac{2\pi R}{R-r} \to 0 \text{ as } n \to \infty$$

or $|U_{n+1}| \le M\left(\frac{r}{R}\right)^{n+1} \frac{1}{1-\frac{r}{R}} \to 0$ as $n \to \infty$

Hence $\lim_{n\to\infty} U_{n+1} = 0$

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$$|V_{n+1}| \le \frac{1}{2\pi}\int_{C'} |f(t)| \left|\frac{t-a}{z-a}\right|^{n+1} \frac{|dt|}{|z|-|t|}$$

$$\le \frac{M'}{2\pi}\left(\frac{R'}{r}\right)^{n+1} \cdot \frac{2\pi R'}{r-R'}$$

$$\text{or } |V_{n+1}| \le M'\left(\frac{R'}{r}\right)^{n+1} \cdot \frac{1}{\frac{r}{R'}-1} \to 0 \text{ as } n\to\infty$$

Hence $V_{n+1}=0$ Making $n\to\infty$ in (c)

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$$

12 रवि

Deduction:

When C_0 is a circle whose equation is $R' < |t-a| = R_0 < R$

then $$a_n = \frac{1}{2\pi i}\int_{C_0} \frac{f(t)\,dt}{(t-a)^{n+1}}$$

$$b_n = \frac{1}{2\pi i}\int_{C_0} \frac{f(t)\,dt}{(t-a)^{-n+1}} = a_{-n}$$

In these case (1) becomes

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28	29	30				

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} (z-a)^{-n} a_{-n}$$

$$= \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=-1}^{-\infty} (z-a)^n a_n$$

$$= \sum_{-\infty}^{\infty} a_n (z-a)^n$$

or $$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$$

where $$a_n = \frac{1}{2\pi i} \int_C \frac{f(t)\,dt}{(t-a)^{n+1}}$$